

# Prospects for gravitational-wave observations of neutron-star tidal disruption in neutron-star/black-hole binaries

Michele Vallisneri

*Theoretical Astrophysics 130-33, California Institute of Technology, Pasadena CA 91125.  
INFN, Sez. di Milano, Gruppo Collegato di Parma/Università di Parma, 43100 Parma, Italy.*  
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For an inspiraling neutron-star/black-hole binary (NS/BH), we estimate the gravity-wave frequency  $f_{\text{td}}$  at the onset of NS tidal disruption. We model the NS as a tidally distorted, homogeneous, Newtonian ellipsoid on a circular, equatorial geodesic around a Kerr BH. We find that  $f_{\text{td}}$  depends strongly on the NS radius  $R$ , and estimate that LIGO-II (ca. 2006–2008) might measure  $R$  to 15% precision at 140 Mpc ( $\sim 1$  event/yr under current estimates). This suggests that LIGO-II might extract valuable information about the NS equation of state from tidal-disruption waves.

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The equation of state of the bulk nuclear matter inside a neutron star (NS) is poorly understood [1]. For example, candidate equations of state that are compatible with nuclear physics experiments and theory predict, for a  $1.4 M_{\odot}$  NS, a radius anywhere from about 8 km to 16 km [2]. Thorne has conjectured that insights into the equation of state might come from measurements of the gravitational waveforms emitted by merging NS/NS binaries and/or tidally disrupting NS's in neutron-star/black-hole (NS/BH) binaries [3,4]. More recently, Newtonian models of NS/NS mergers have given strong evidence that the merger waves *do* carry equation-of-state information, but for NS/NS are emitted at frequencies ( $\sim 1400$ – $2800$  Hz) too high for measurement by LIGO-type gravity-wave interferometers [5,6]. In this paper, we show that the prospects for NS/BH measurements are much brighter.

Central to these prospects is the question of whether NS tidal-disruption waves lie in the band of good interferometer sensitivity (for LIGO-II,  $\sim 30$ – $1000$  Hz [7]; see Fig. 1). Numerical modeling of NS tidal disruption in NS/BH binaries is only now getting underway [8] and has not yet included computations of the emitted gravity waves or even their frequency bands. As a result, the best frameworks now available for estimating the tidal-disruption gravity-wave band are highly simplified, quasi-analytic models by Shibata [9] and by Wiggins and Lai [10], which represent the inspiraling NS as an irrotational [11], incompressible or polytropic Newtonian ellipsoid, moving on a circular, equatorial geodesic orbit around a Kerr BH, and being tidally distorted by the Kerr Riemann tensor. For simplicity we focus on Shibata's homogeneous models, and then appeal to the polytropic models for evidence that compressibility has only small

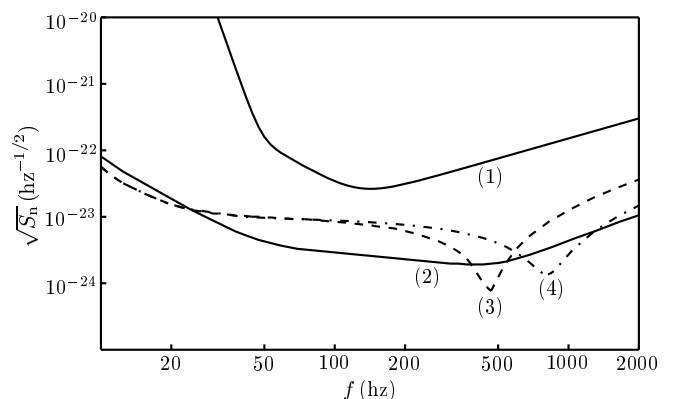


FIG. 1. Plot of (square root of) noise spectral density for different LIGO configurations: (1) LIGO-I; (2) LIGO-II wide-band; (3), (4) LIGO-II narrowband centered on 500 and 850 Hz. Curves (1)–(3) are from [7]; curve (4) was produced by K. Strain using the same detector specifications as in [7].

effects.

In Shibata's analysis, the NS gravitational field, its centrifugal potential, and the Newtonian tidal potential constructed from the Kerr Riemann tensor are all quadratic functions of position. As a result, a class of equilibrium solutions are the classic irrotational, homogeneous *Roche-Riemann ellipsoids* [12]. Given a choice of the binary parameters  $M$ ,  $a$  and  $r$  (the BH mass and angular momentum per unit mass, and the orbital separation, i. e., the Boyer-Lindquist radius of the geodesic), there is a one-parameter family of such NS models with density  $\rho$  ranging downward through the family to a minimum  $\rho_{\text{cr}}(M, a, r)$ .

We model the inspiraling NS as one of Shibata's irrotational ellipsoids, identified by its mass  $m$ , and its density  $\rho$  or mean radius  $R = (3m/4\pi\rho)^{1/3}$ . In our simple framework, the uncertainty in  $m(R)$  embodies the uncertainty about the NS equation of state. We describe the inspiral as a sequence of circular, equatorial Kerr geodesics that shrink inward until the NS reaches the innermost stable circular orbit,  $r = r_{\text{isco}}$ , or begins to tidally disrupt [which happens at the radius  $r_{\text{td}}$  where the star's density  $\rho$  matches the critical density  $\rho_{\text{cr}}(M, a, r_{\text{td}})$ ].

The Kerr geometry provides a one-to-one correspondence between the orbital radius  $r_{\text{td}}$  and the gravity-wave frequency  $f_{\text{td}}$  at which tidal disruption begins:

$$f_{\text{td}}(M, a, r_{\text{td}}) = \frac{1}{\pi(a + \sqrt{r_{\text{td}}^3/M})} \quad (1)$$

(here and below we set  $G = c = 1$ ). It is this  $f_{\text{td}}$  that LIGO-II can measure. Having measured  $f_{\text{td}}$  and determined the masses  $M$  and  $m$  from the observed inspiral waveforms [13], one can compute  $r_{\text{td}}$  and thence the NS density  $\rho = \rho_{\text{cr}}(M, a, r_{\text{td}})$  and the mean NS radius  $R$ . Thereby, the LIGO-II observations can determine a point on the NS mass-radius curve  $m(R)$ , which represents the NS equation of state in our simplified analysis. Even one such point could give valuable information about the *real* NS equation of state, and several such points could determine it remarkably well [14]. Moreover, the tidal-disruption waveforms observed for a single NS might carry much more information about NS structure than that in the single measured point  $(m, R)$ .

To estimate the accuracy with which LIGO-II might determine the NS radius  $R$ , we need the explicit relationship between  $R$  and the disruption-onset frequency  $f_{\text{td}}$ . More precisely, we need  $R(m, M, a, f_{\text{td}})$ , which can be derived as follows: (i)  $r_{\text{td}}(M, a, f_{\text{td}})$  is obtained by inverting Eq. (1); (ii)  $\rho_{\text{cr}}(M, a, r_{\text{td}})$  is obtained by solving Eq. (3.9) of [9] for the ratios of semiaxes of the equilibrium configurations, and then extremizing Eq. (3.10) of [9], in which  $\tilde{\Omega}^2 = M/(\pi\rho r^3)$ ; (iii) then  $R$  is obtained as  $R = [3m/4\pi\rho_{\text{cr}}(M, a, r_{\text{td}})]^{1/3}$ . The result has the form

$$R(m, M, a, f_{\text{td}}) = m^{1/3} M^{2/3} \hat{D}\left[\frac{a}{M}, f_{\text{td}} M\right], \quad (2)$$

where  $\hat{D}$  is a dimensionless function with remarkably weak dependence on  $a/M$  [15]. This  $R(f_{\text{td}})$  is shown in Fig. 2 for various  $M$ , for  $a/M = 0.998$  (the curves for other  $a/M$  are almost identical to these), and for  $m = 1.4 M_{\odot}$  [16]. The radii shown,  $R = 8$ – $16$  km for  $m = 1.4 M_{\odot}$ , correspond to the range of predictions by plausible NS equations of state [2]. The curves in Fig. 2 are well approximated by the formula (with  $G = c = 1$ )

$$\frac{R}{m^{1/3} M^{2/3}} \approx \begin{cases} 0.145 (f_{\text{td}} M)^{-0.71} & \text{for } f_{\text{td}} M \lesssim 0.04, \\ 0.069 (f_{\text{td}} M)^{-0.95} & \text{for } f_{\text{td}} M \gtrsim 0.04. \end{cases} \quad (3)$$

Although the BH spin parameter  $a$  has negligible influence on the function  $R(f_{\text{td}})$ , it strongly influences the radius  $r_{\text{isco}}$  of the innermost stable circular orbit [17]. If the NS is still intact when it reaches  $r_{\text{isco}}$ , it then will plunge rapidly into the BH and the tidal-disruption waves, if any, will likely be so weak and short-lived as to be useless for measuring NS properties. Thus, there is not much hope of measuring tidal disruption unless  $f_{\text{td}} < f_{\text{plunge}} = [\text{Eq. (1) with } r_{\text{td}} \text{ replaced by } r_{\text{isco}}(M, a)]$ ; i. e., unless  $f_{\text{td}}$  is left of the relevant big dot in Fig. 2.

Fig. 2 and the above discussion show that (i) for a wide range of realistic parameter values, the tidal disruption

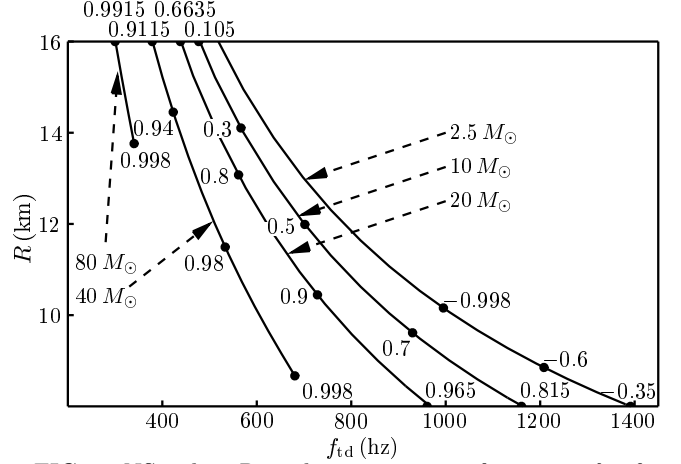


FIG. 2. NS radius  $R$  vs. disruption-onset frequency  $f_{\text{td}}$ , for  $m = 1.4 M_{\odot}$  and  $M = 2.5, 5, 10, 20, 40, 80 M_{\odot}$ . The black dots, parametrized by  $a/M$ , specify the onset of plunge into the BH; tidal disruption is measurable only for  $f_{\text{td}}$  left of the plunge point, i. e., for  $R$  above it. Negative  $a/M$  indicates retrograde geodesic orbits.

will occur before the plunge begins, and (ii) for all realistic parameters except a very narrow range ( $M \lesssim 10 M_{\odot}$  and  $R \lesssim 10$  km), the tidal-disruption waves will fall in the range of good LIGO sensitivity,  $f \lesssim 1000$  Hz. The Lai-Wiggins polytropic NS models [10] give similar curves and conclusions: for polytropic indices  $n = 0.5$  and  $1.0$ , which approximate NS equations of state, the  $R(f_{\text{td}})$  curves are displaced upward in frequency from those of Fig. 2 by a mere  $\sim 50$  and  $\sim 100$  Hz.

Turn now to an estimate of the accuracy to which LIGO-II could measure  $f_{\text{td}}$  (and thence the NS radius  $R$ ) using *Wiener optimal filtering* [18,19]. The measured gravity-wave data stream  $g(t)$  is compared to a set of theoretical inspiral *templates*  $h(\theta^i; t)$ , indexed by the parameters  $\theta^i$  of the binary; a “best fit”  $\hat{\theta}^i$  is found which maximizes the likelihood of observing  $g(t)$  given a “true” signal  $h(\hat{\theta}^i; t)$ , and given a statistical model of the detector noise [a Gaussian [20] random process with zero mean and spectral density  $S_n(f)$ ]. For strong enough signals,  $\hat{\theta}^i$  will have a gaussian distribution centered around its “true value”  $\tilde{\theta}^i$ , with covariance matrix [19]

$$C_{ij} = (\Gamma^{-1})_{ij}, \quad \Gamma_{ij} \approx 2 \left\langle \frac{\partial h}{\partial \theta^i}(\hat{\theta}^k) \left| \frac{\partial h}{\partial \theta^j}(\hat{\theta}^k) \right. \right\rangle, \quad (4)$$

where the “inner product”  $\langle \dots \rangle$  is defined for any two real data streams  $g(t)$ ,  $h(t)$  in terms of their Fourier transforms  $\tilde{g}(f)$ ,  $\tilde{h}(f)$  by

$$\langle g, h \rangle = \int_{-\infty}^{\infty} df \frac{\tilde{g}(f) \tilde{h}^*(f)}{S_n(|f|)}. \quad (5)$$

Because so little is known about the tidal disruption and our NS models are so crude, we use the simplest

of templates in our analysis: slow-motion, quadrupolar waveforms for point particles in circular, Keplerian orbits with quadrupole-governed inspiral. The Fourier-transformed waveform, squared and averaged over binary directions and orientations, is given by [21]

$$\langle |\tilde{h}_b|^2 \rangle = \frac{\pi}{30} \frac{\mu M_T^3}{d^2} \frac{1}{(\pi M_T f)^{7/3}} \theta(f_{\text{plunge}} - f), \quad (6)$$

where  $\mu$  and  $M_T$  are the reduced and total masses,  $d$  is the distance to the binary, and the step function shuts the signal off at the onset of plunge.

For typical observations, optimal filtering of the inspiral signal should give good estimates of  $M$  and  $m$  [13]. We therefore assume that the accuracy in measuring  $R$  is limited solely by the uncertainty of  $f_{\text{td}}$  [22]. The estimation of  $f_{\text{td}}$  depends heavily on the details of the tidal-disruption waveforms, which are largely unknown. However, it is reasonable to expect tidal disruption to be a sudden event that significantly weakens gravity-wave emission within a few dynamical time-scales after  $f_{\text{td}}$  has been reached [23]. Correspondingly, we employ a toy model where the inspiral waveform of Eq. (6) dies out over a frequency band ( $f_{\text{td}}, f_{\text{td}} + \delta f$ ):

$$\tilde{h}_{\text{td}}(f) = \begin{cases} \tilde{h}_b(f) & \text{if } f < f_{\text{td}}, \\ \tilde{h}_b(f) \Theta(\frac{f-f_{\text{td}}}{\delta f}) & \text{if } f_{\text{td}} < f < f_{\text{td}} + \delta f, \\ 0 & \text{if } f > f_{\text{td}} + \delta f, \end{cases} \quad (7)$$

where  $\Theta(x) = 1 - x$  (*linear decay*), or  $\Theta(x) = 10^{-x}$  (*exponential decay*). The standard deviation of the “best fit”  $\hat{f}_{\text{td}}$  is given by Eq. (4) as  $\Delta \hat{f}_{\text{td}} = [\Gamma_{f_{\text{td}} f_{\text{td}}}(\tilde{h}_{\text{td}})]^{-1/2}$ .

We have evaluated  $\Delta \hat{f}_{\text{td}}$  numerically, using the signal model from Eqs. (6), (7) and the inner product (5) with the LIGO-II noise curves  $S_n(r)$  of Fig. 1. We have then computed the  $2\sigma$  range of the NS radii  $R$  from the relation  $R_{\pm} = R(m, M, a, \hat{f}_{\text{td}} \mp 2\Delta \hat{f}_{\text{td}})$  [Eq. (2)]. The uncertainty in  $R$ , defined as  $\Delta R = (R_+ - R_-)/2$ , scales roughly linearly with  $d$  [because  $\Delta \hat{f}_{\text{td}}$  is proportional to  $d$  through Eqs. (4), (6)], and is quite sensitive to the choice of the shut-off model [it scales roughly as  $(\delta f)^{1/2}$  and is lower for the exponential decay than for the linear one]. In Table I we report the fractional uncertainty  $\Delta R/R$ , averaged over the range  $10 \text{ km} < R < 15 \text{ km}$ , for choices of parameters motivated by the following.

The NS mass  $m$  was set to be  $1.4 M_{\odot}$  [16]. The distance  $d$  and the BH masses  $M$  were chosen to represent two different scenarios: (i) low-mass BH’s, with  $M = 2.5 M_{\odot}$  at 65 Mpc ( $\sim$  one merger/yr according to Bethe and Brown [24]); (ii) higher mass BH’s, with  $M = 10, 20$ , and  $40 M_{\odot}$ , at 140 Mpc (massive main-sequence binaries are thought to produce NS/BH binaries with  $M \sim 10 M_{\odot}$  and coalescence rates up to  $\sim$  one event/yr out to 140 Mpc, but possibly much less [25]; capture NS/BH binaries formed in globular clusters might have  $M$  as large as hundreds of  $M_{\odot}$  [26], but with exceedingly uncertain rates). Finally, we considered three different gravity-wave shut-off

TABLE I. Fractional uncertainty  $\Delta R/R$ , averaged over the range  $10 \text{ km} < R < 15 \text{ km}$ . The rows correspond to different BH masses, the columns to different gravity-wave decay and detector noise models [labeled by (2)–(4) in accordance with Fig. 1]. Estimation of the NS radius is considered insignificant and no quote is given if  $\Delta R/R > 25\%$ .

$M \setminus S_n$	$\Delta R/R$ (%) for $10 \text{ km} < R < 15 \text{ km}$											
	lin. decay			exp. decay			e. d., $n = 1$			lin. decay		
	$\delta f = f_{\text{td}}/6$			$\delta f = f_{\text{td}}/2$			$\delta f = f_{\text{td}}/2$			$\delta f = f_{\text{td}}/2$		
	(2)	(3)	(4)	(2)	(3)	(4)	(2)	(3)	(4)	(2)	(3)	(4)
$2.5 M_{\odot}^{\text{a}}$	12	18 <sup>c</sup>	8	20	–	13	24 <sup>c</sup>	–	25	21 <sup>c</sup>	–	17
$10 M_{\odot}^{\text{b}}$	14	17 <sup>c</sup>	10	23	–	17	25 <sup>c</sup>	–	21	25 <sup>c</sup>	–	19
$20 M_{\odot}^{\text{b}}$	10	16	10	16	14 <sup>c</sup>	16	23	–	14	22	25 <sup>c</sup>	16
$40 M_{\odot}^{\text{b}}$	7	6	11	11	10	19	20	10 <sup>c</sup>	20	17	19	23

<sup>a</sup>At 65 Mpc. <sup>b</sup>At 140 Mpc. <sup>c</sup>For  $12 \text{ km} < R < 15 \text{ km}$ .

models: (i) an optimal-precision model with linear decay and  $\delta f = f_{\text{td}}/6$  (the lower limit set by the uncertainty principle on the frequency spread of waves emitted during 3 orbital periods, supposedly a typical time-scale for complete disruption [23]); (ii) a fiducial model with exponential decay and  $\delta f = f_{\text{td}}/2$  (this scaling is supported by the gravitational waveforms computed by Zhuge *et al.* [5] for tidal disruption in the final stages of NS/NS inspirals); this model was also used to evaluate errors for  $n = 1$  compressible polytropes; (iii) a conservative model with linear decay and  $\delta f = f_{\text{td}}/2$ .

The estimates for our fiducial decay model suggest that  $R$  may be determined with a precision of  $\sim 15\%$  using the 850 Hz-narrowband LIGO-II configuration [curve (4) of Fig. 1], and with a somewhat worse precision for wide-band LIGO-II [curve (2)]. If the optimal-precision decay model is correct, the error might be as low as  $\sim 6\text{--}10\%$ . The usefulness of the 500 Hz-narrowband interferometer [curve (3) of Fig. 1] is limited to the heavier BH’s or to the larger NS’s, which have lower  $f_{\text{td}}$ . Our estimates are inferior for the Lai-Wiggins compressible polytropes [10], examined in the least favorable case ( $n = 1$ ), and for the most conservative decay model; even then, an 850 Hz-narrowband LIGO-II might be able to provide significant information about  $R$ .

The accuracy of our analysis is limited by several factors. Sources of error in the frequency  $f_{\text{td}}(m, M, a, R)$  at which tidal disruption begins to significantly change the inspiral waveforms include: (i) the use of the test-mass approximation for the NS orbit, when actually  $m \not\ll M$ , especially for the low-mass Bethe-Brown case; (ii) the use of the Riemann tensor to compute tidal-disruption forces when the NS diameter is not, typically, small compared to the distance from the NS center to the horizon [27]; (iii) the idealization of the NS as a homogeneous or polytropic ellipsoid; (iv) the fact that the point at which the observed waveforms show a clear deviation from standard inspiral may actually come a few orbits earlier (due to tidal coupling) or later than  $f_{\text{td}}$ .

Moreover, the shutoff of the inspiral waves might differ strongly from the *ad hoc*, idealized form (7) that we employed, and the actual tidal-disruption waveforms might contain features that carry more accurate and useful information about the equation of state than the information in  $f_{\text{td}}$ . For example, for the final stage of NS/NS inspirals, Zhuge *et al.* [5] predict a gravity-wave cutoff followed by a valley and then by a moderately sharp peak, which is quite different from a simple cutoff.

Given these large uncertainties, our results can only be rough indications of the prospects for learning about NS's from tidal-disruption waveforms. They do, however, suggest that observations of tidal disruption in NS/BH binaries might be possible in  $\sim 2006$ – $2008$  with second generation LIGO interferometers, and that these observations may yield useful insights into the NS equation of state. The success of this endeavor will require the development of better theoretical and numerical techniques for modeling NS tidal disruption and computing the dependence of the disruption waveforms on the NS equation of state; we strongly advocate such an effort. It would be remarkable if a better understanding of NS's, which are essentially giant nuclei, came from observations of NS tidal disruption, i. e., *fission at a stellar scale*—just as we learned about *atomic* nuclei by observing their violent disruption.

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[22] For a signal to noise ratio  $\gtrsim 10$  (fairly typical of the observations examined in this paper), and if spins can be treated as negligible,  $\Delta m/m$ ,  $\Delta M/M \lesssim 0.02$  [13], and from Eq. (3) the influence of  $\Delta m$  and  $\Delta M$  on  $\Delta R$  gives  $\Delta R/R \sim 0.005$ . If spins are important these errors increase tenfold, but might be considerably reduced if the *a priori* knowledge of  $m$  from known NS/NS binaries [16] can be applied to NS/BH systems.  
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